

A Supersymmetric Treatment of a Particle Subjected to a Ring-shaped Potential

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Abstract

The ring-shaped Hartmann potential $V = \eta\sigma^2\epsilon_0 \left(\frac{2a_0}{r} - \frac{\eta a_0^2}{r^2 \sin^2\theta} \right)$ was introduced in quantum chemistry to describe ring-shaped molecules like benzene. In this article, fundamental concepts of supersymmetric quantum mechanics (SUSYQM) are discussed. The energy eigenvalues and (radial) eigenfunctions of the Hartmann potential are subsequently rederived using the techniques of SUSYQM.

Key words: supersymmetry, Hartmann potential, supersymmetric quantum mechanics, ring-shaped potential, superpotential

1 Introduction

In 1972, an exactly solvable ring-shaped potential was introduced by H. Hartmann [1]. The Hartmann potential is given by the following expression,

$$V = \eta\sigma^2\epsilon_0 \left(\frac{2a_0}{r} - \frac{\eta a_0^2}{r^2 \sin^2\theta} \right) \quad (1)$$

where

$$a_0 = \frac{\hbar^2}{\mu e^2} \quad \text{and} \quad \epsilon_0 = -\frac{1}{2} \frac{\mu e^4}{\hbar^2} \quad (2)$$

μ is the particle mass, η and σ are positive real parameters which range from about 1 to 10 in theoretical chemistry applications [2] and r, θ are in spherical coordinates. Following the exact solution of the Schrödinger equation given by Hartmann [1], alternative methods of solutions in spherical coordinates [3] and squared parabolic coordinates [4, 5, 6, 7, 8] had been given. In this article, the author presents another alternative method of solution in spherical coordinates using supersymmetry (SUSY).

The concept of supersymmetry (SUSY) has been used in particle physics in the past two decades [9, 10]. It was discovered in 1971 by Gel'fand and Likhtman [11]. Simply put, supersymmetry is a symmetry which relates fermionic and bosonic degrees of freedom. At present, particle physicists believe that it is an essential ingredient in unifying the four fundamental forces in nature namely the electromagnetic, weak, strong and gravitational interactions.

Supersymmetric theories of the four fundamental interactions entail the presence of SUSY partners which have the same mass as their corresponding ordinarily observed particles.¹ Unfortunately, these have not been observed in nature. To make sense out of this experimental fact, theorists believe that SUSY must be “broken” at ordinary energies. The search for a mechanism to break SUSY led Witten [12] in 1981 to study SUSY breaking in the simplest case of SUSY quantum mechanics. In fact, studies in SUSYQM during its early years, were confined solely for understanding SUSY breaking.

However, it was eventually discovered that SUSYQM can have interesting applications besides its use in the study of SUSY breaking. At present, it has found its way in many areas of physics including atomic physics, statistical physics, nuclear physics. etc. [13]. Through the present article, the author hopes to contribute to the utilization of SUSYQM in theoretical chemistry.

¹For example, the SUSY partner of the electron is called a selectron while that of a photon is a photino.

One simple use of SUSYQM is in obtaining the eigenvalues and eigenfunctions of the Schrödinger equation [14, 15]. A two-dimensional supersymmetric solution of the Hartmann potential in “square” parabolic coordinates [4] had been proposed. In the present article, the author formulates a one-dimensional SUSY solution similar to that of the hydrogen atom [14]. This avoids the complications of a two-dimensional SUSYQM formulation. Using SUSY, it is shown that the eigenvalues and (radial) eigenfunctions can be obtained. The techniques employed here are largely due to Schwabl [14].

In section 2, a pedagogical introduction to SUSYQM is developed. Only the concepts and equations which are essential for the present paper are presented.

Section 3 gives a detailed account of how SUSYQM can be used to obtain the eigenvalues and (radial) eigenfunctions of the Hartmann potential. The discussion is heavily supplemented by energy level diagrams to facilitate the understanding of the hierarchy of hamiltonians and how the SUSY operators A_L^+ are used to obtain the eigenfunctions.

Some conclusions are given in section 4.

2 Supersymmetric Quantum Mechanics

As mentioned in the Introduction, SUSY was first applied to particle physics, whose language is quantum field theory. In quantum field theory, a particle is represented by a component field φ_i and its dynamics is described by a lagrangian density $\mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$ where $[\partial_\mu \equiv (\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla})]$. The word “supersymmetry” was originally used to describe the symmetry which transforms a field φ to another field ψ whose intrinsic spin differs from φ by $\frac{1}{2}\hbar$. In SUSYQM, which will be described here, we will use the term “supersymmetry” in a more general sense. It will be used to denote systems which can be described by the SUSY algebra in supersymmetric field theory.

SUSYQM [12, 16] is characterized by the existence of the charge operators Q_i , where $i = 1, 2, \dots, N$ such that they obey the SUSY algebra (denoted by $\text{sqm}(N)$),

$$\{Q_i, Q_j\} = \delta_{ij} H_{ss} \quad [Q_i, H_{ss}] = 0 \quad (3)$$

where H_{ss} is the supersymmetric Hamiltonian, $\{ \}$ and $[\]$ are anticommutator and commutator respectively. We consider only $\text{sqm}(2)$ with charge operators Q_1 and Q_2 and construct the linear combinations

$$Q = \frac{1}{\sqrt{2}} (Q_1 + iQ_2) \quad \text{and} \quad Q^\dagger = \frac{1}{\sqrt{2}} (Q_1 - iQ_2) . \quad (4)$$

From equations 3 and 4, The SUSY algebra is then

$$\{Q, Q^\dagger\} = H_{ss}, \quad Q^2 = 0, \quad (Q^\dagger)^2 = 0 \quad (5)$$

with

$$[Q, H_{ss}] = 0 \quad \text{and} \quad [Q^\dagger, H_{ss}] = 0. \quad (6)$$

The above SUSY algebra can be realized by letting

$$Q = \begin{bmatrix} 0 & 0 \\ A^- & 0 \end{bmatrix} ; \quad Q^\dagger = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} \quad (7)$$

where

$$(A^-)^\dagger = A^+. \quad (8)$$

From equations 5 and 7, the supersymmetric hamiltonian is

$$H_{ss} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad (9)$$

where

$$H_1 = A^+ A^- \quad \text{and} \quad H_2 = A^- A^+. \quad (10)$$

The hamiltonians H_1 and H_2 are said to be “supersymmetric” partners of each other.

The hamiltonian of the Schrödinger equation can always be factorized in the form of equation 10. Consider the hamiltonian [13]

$$H_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x) \quad (11)$$

such that

$$H_1 \psi_{(1)}^n = \left[-\frac{1}{2} \frac{d^2}{dx^2} + V_1(x) \right] \psi_{(1)}^n = E_{(1)}^n \psi_{(1)}^n \quad (12)$$

where $V_1(x)$ is chosen such that the ground state $\psi_{(1)}^0$ has energy equal to zero. The hamiltonian in equation 11 can be put in the form of equation 10 by letting

$$A_1^- = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + W_1 \right) \quad \text{and} \quad A_1^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W_1 \right) \quad (13)$$

provided that the “superpotential” W_1 satisfies the Ricatti equation

$$V_1(x) = \frac{1}{2} \left[W_1^2 - \frac{dW_1}{dx} \right]. \quad (14)$$

As long as equation 14 has a solution W_1 , the one-dimensional Schrödinger equation can be made supersymmetric by the construction given in equations 13, 10 and 9. The challenge then in using SUSYQM techniques is not in the mechanics of the construction of just any SUSY hamiltonian, but in finding a suitable superpotential (or $V_1(x)$) to construct a SUSY hamiltonian which will be relevant to the problem at hand. It is a common practice to choose or pose as an ansatz the W_1 to solve a physical problem [14, 17].

The SUSY partner of H_1 , namely H_2 is then given by

$$H_2 = -\frac{1}{2} \frac{d^2}{dx^2} + V_2(x) = A_1^- A_1^+ \quad (15)$$

where

$$V_2(x) = \frac{1}{2} \left[W_1^2 + \frac{dW_1}{dx} \right]. \quad (16)$$

Note that H_2 is altogether a new hamiltonian. An astute reader will immediately realize that one can repeat the procedure in constructing H_2 from H_1 to construct an H_3 from H_2 such that

$$H_2 = A_2^+ A_2^- \quad (17)$$

with

$$A_2^- = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + W_2 \right) \quad \text{and} \quad A_2^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W_2 \right) \quad (18)$$

and with a new Ricatti equation

$$V_2(x) = \frac{1}{2} \left[W_2^2 - \frac{dW_2}{dx} \right]. \quad (19)$$

W_2 in equation 19 is then solved to construct A_2^\pm . H_3 can then be constructed as

$$H_3 = -\frac{1}{2} \frac{d^2}{dx^2} + V_3(x) = A_2^- A_2^+ \quad (20)$$

where

$$V_3(x) = \frac{1}{2} \left[W_2^2 + \frac{dW_2}{dx} \right]. \quad (21)$$

We can evidently construct a “hierarchy” of SUSY-partner hamiltonians $H_1, H_2, H_3, \dots, H_n$ starting from H_1 .

Let us go back to the first two hamiltonians we started with namely H_1 and H_2 . Since $V_1(x)$ in equation 11 is chosen such that its ground state wave function $\psi_{(1)}^0$ has eigenvalue equal to zero, equation 10 gives

$$H_1 \psi_{(1)}^0 = 0 \quad \implies \quad A_1^+ A_1^- \psi_{(1)}^0 = 0. \quad (22)$$

Equation 22 implies

$$A_1^- \psi_{(1)}^0 = 0. \quad (23)$$

With equation 13 and knowing W_1 , equation 23 allows one to calculate the ground state of H_1 by solving the resulting first order differential equation.

Consider any eigenstate of H_1 , $\psi_{(1)}^n$ with energy $E_{(1)}^n$. We have

$$H_1 \psi_{(1)}^n = E_{(1)}^n \psi_{(1)}^n \quad (24)$$

or from equation 10

$$A_1^+ A_1^- \psi_{(1)}^n = E_{(1)}^n \psi_{(1)}^n. \quad (25)$$

Applying A_1^- to equation 25, $A_1^- A_1^+ (A_1^- \psi_{(1)}^n) = E_{(1)}^n A_1^- \psi_{(1)}^n$ or with equation 15

$$H_2 (A_1^- \psi_{(1)}^n) = E_{(1)}^n (A_1^- \psi_{(1)}^n). \quad (26)$$

Conversely, consider an eigenfunction $\psi_{(2)}^n$ of H_2 with eigenvalue $E_{(2)}^n$. With equation 15, we get $H_2 \psi_{(2)}^n = A_1^- A_1^+ \psi_{(2)}^n = E_{(2)}^n \psi_{(2)}^n$. Multiplying by A_1^+ , we have, $A_1^+ A_1^- (A_1^+ \psi_{(2)}^n) = E_{(2)}^n A_1^+ \psi_{(2)}^n$. With equation 10

$$H_1 (A_1^+ \psi_{(2)}^n) = E_{(2)}^n (A_1^+ \psi_{(2)}^n). \quad (27)$$

Equations 26 and 27 imply that the hamiltonians H_1 and H_2 have identical eigenvalues except for the ground state $\psi_{(1)}^0$ of H_1 (since $A_1^- \psi_{(1)}^0 = 0$ in equation 26 and this is unnormalizable). In addition, we can see that if you know an eigenfunction of H_1 , i.e. $\psi_{(1)}^n$, then an eigenfunction $A_1^- \psi_{(1)}^n$ of H_2 can be formed. Similarly, an eigenfunction $A_1^+ \psi_{(2)}^n$ of H_1 can be formed given an eigenfunction $\psi_{(2)}^n$ of H_2 . The preceding analysis can then be extended to the hierarchy of hamiltonians discussed earlier. These observations are illustrated in figure 1 [14, 16].

Herein lies a very important consequence of SUSYQM. The energy eigenfunctions of the hierarchy of hamiltonians are related by the SUSY operators A^\pm . If one knows the eigenvalues and eigenfunctions of a particular H_n , then one can get the eigenvalues and eigenfunctions of its SUSY partner.

Note that what we have discussed is SUSYQM in one dimension. There had been work done in doing SUSYQM in two or more dimensions [4, 18]. To be able to apply one dimensional SUSYQM to the Hartmann potential, we will do a separation of variables of the resulting Schrödinger equation. To be able to form a hamiltonian from a separated one dimensional differential equation that can be an element of an H_{ss} , this one dimensional differential equation must be of the form of equation 12 (no first derivative term) and must yield an infinite tower of states as for H_n of figure 1 [19]. As we will see, upon separation of variables for the Hartmann potential, only the radial equation yields an interesting SUSY.

3 Supersymmetry in the Hartmann Hamiltonian

With the SUSYQM concepts introduced in section 2, we are now ready to obtain the eigenvalues and radial eigenfunctions of the Hartmann potential in spherical coordinates.

The Schrödinger equation in spherical coordinates for a particle of mass μ subjected to the Hartmann potential in equation 1 is given by

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \left[\frac{2\eta\sigma^2\epsilon_0 a_0}{r} - \frac{\eta^2\sigma^2 a_0^2\epsilon_0}{r^2 \sin^2\theta} \right] \psi = E\psi . \quad (28)$$

Assuming a solution

$$\psi = R(r)\Theta(\theta)\Phi(\phi) \quad (29)$$

equation 28 can be separated into three differential equations [1]

$$\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = -m^2 \quad (30)$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) - \left(\frac{M^2}{\sin^2\theta} - L(L+1) \right) \Theta = 0 \quad (31)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - L(L+1) \frac{R}{r^2} + \frac{8\pi^2\mu}{\hbar^2} \left(E + \frac{\eta\sigma^2 e^2}{r} \right) R = 0 \quad (32)$$

where

$$M^2 = m^2 + \eta^2\sigma^2 . \quad (33)$$

Looking at equations 30 to 32, we realize that these closely resemble the separated equations of the hydrogen atom [20]. As shown by reference [19], the only interesting separable SUSY in the hydrogen atom in spherical coordinates results from the radial equation. Their argument is as follows. Looking at equation 30, and comparing it with equation 12, we see that $V_1 = 0$. Equation 31 on the other hand can be cast to a form similar to equation 12 by multiplying it by a modulation factor $\frac{f(\cos\theta)}{[1-\cos^2\theta]^{1/2}}$. The eigenvalue of $f(\cos\theta)$ is zero and no infinite tower of states can be generated. Hence, the Φ and Θ solutions cannot be given by SUSYQM. They can be solved by conventional means [1]

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \dots \quad (34)$$

$$\Theta(\theta) = \mathcal{P}_L^{|M|}(\cos\theta), \quad L = \nu' + |M|, \quad \nu' = 0, 1, 2, \dots \quad (35)$$

where $\mathcal{P}_L^{|M|}(\cos\theta)$ reduces to the associated Legendre polynomials when $\eta\sigma \rightarrow 0$.

The radial equation 32 can be cast into a form similar to 12 by letting

$$R = \frac{u}{r}. \quad (36)$$

Substituting equation 36 into equation 32, yields

$$H_L u = \left[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{L(L+1)}{2r^2} - \frac{\gamma}{r} \right] u = \frac{\mu E}{\hbar^2} u \quad (37)$$

with

$$\gamma \equiv \frac{\mu \eta \sigma^2 e^2}{\hbar^2}. \quad (38)$$

Equation 37 is similar to that of the hydrogen atom's radial equation. We thus claim that we can obtain the eigenvalues and radial eigenfunctions by looking at the hamiltonian [14]

$$\mathcal{H}_L = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{L(L+1)}{2r^2} - \frac{\gamma}{r} + \frac{1}{2} \left(\frac{\gamma}{L+1} \right)^2 \quad (39)$$

which yields a Ricatti equation (from equations 39, 11 and 14)

$$\frac{L(L+1)}{2r^2} - \frac{\gamma}{r} + \frac{1}{2} \left(\frac{\gamma}{L+1} \right)^2 = \frac{1}{2} \left[W_L^2 - \frac{dW_L}{dr} \right] \quad (40)$$

whose solution is

$$W_L = -\frac{L+1}{r} + \frac{\gamma}{L+1}. \quad (41)$$

Equation 41 and 13 yield

$$A_L^\pm = \frac{1}{\sqrt{2}} \left(\mp \frac{d}{dr} - \frac{L+1}{r} + \frac{\gamma}{L+1} \right). \quad (42)$$

From equation 42 and 15, we construct the SUSY-partner hamiltonian of \mathcal{H}_L in equation 39,

$$\mathcal{H}_{L+1} = A_L^- A_L^+ = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{(L+1)(L+2)}{2r^2} - \frac{\gamma}{r} + \frac{1}{2} \left(\frac{\gamma}{L+1} \right)^2. \quad (43)$$

Comparing equations 37 and 39 and with equation 43, we realize that

$$\mathcal{H}_L = H_L + \frac{1}{2} \left(\frac{\gamma}{L+1} \right)^2 \quad (44)$$

$$\mathcal{H}_{L+1} = H_{L+1} + \frac{1}{2} \left(\frac{\gamma}{L+1} \right)^2. \quad (45)$$

Let us now start to build up the radial eigenfunctions and in the process get the eigenvalues. Given an $|M|$ value, the lowest L value is $L = |M|$, as can be seen in

equation 35. It is apparent from equations 44 and 45 that we can build the states of the hierarchy of hamiltonians as in figure 1. This is illustrated in figure 2.

Since $\mathcal{H}_{|M|}$ ($\mathcal{H}_{|M|+1}$) and H_L (H_{L+1}) differ only by a constant, (see equations 44 and 45) every eigenfunction of $\mathcal{H}_{|M|}$ ($\mathcal{H}_{|M|+1}$) will be an eigenfunction of H_L (H_{L+1}). Hence, all we have to do is to solve for the eigenfunctions of \mathcal{H}_L . The actual energy for H_L can be found by letting H_L act on the eigenfunctions of \mathcal{H}_L .

For an arbitrary L , equations 23 and 42 give, for the ground states of \mathcal{H}_L , $\psi_{(L)}^0$, the first order differential equation

$$\frac{1}{\sqrt{2}} \left(\frac{d}{dr} - \frac{L+1}{r} + \frac{\gamma}{L+1} \right) \psi_{(L)}^0 = 0 \quad (46)$$

which can easily be solved as

$$\psi_{(L)}^0 = \mathcal{N}_L r^{L+1} \exp(-\kappa_L r) \quad (47)$$

where

$$\kappa_L \equiv \frac{\gamma}{L+1} . \quad (48)$$

Since L is arbitrary here, we realize that equation 47 is the expression for the eigenfunction for the lowest rung (i.e. the ground state) of the tower of states for each of the hamiltonians in figure 2. Since they are also eigenfunctions of H_L , we can write

$$u_L = \mathcal{N}_L r^{L+1} \exp(-\kappa_L r) \quad (49)$$

where we illustrate them in figure 3.

To get the actual energy, we let H_L of equation 37 act on equation 49.

$$H_L u_L = \left[-\frac{1}{2} \frac{d^2}{dr^2} + \left(\frac{L(L+1)}{2r^2} - \frac{\gamma}{r} \right) \right] u_L = \frac{\mu E_L}{\hbar^2} u_L . \quad (50)$$

After some simplification, this yields

$$E_L = -\frac{\Lambda}{(L+1)^2} \quad \Lambda = \eta^2 \sigma^4 |\epsilon_0| . \quad (51)$$

We characterize the energy by the L quantum number for the moment. From equation 51, we can label the energy levels of figure 3 as in figure 4.

It is apparent from figure 4 that we have to label the solutions u as $u_{|M|+1, |M|}$; $u_{|M|+2, |M|+1}$; $u_{|M|+3, |M|+2}$; ... due to the energy of the states. Knowing the eigenstates at the lowest rung of the hierarchy of hamiltonians, $u_{|M|+1, |M|}$; $u_{|M|+2, |M|+1}$; $u_{|M|+3, |M|+2}$;

etc., we can determine the other states by the action of A_L^+ on these eigenstates as in figure 1. This is illustrated in figure 5.

Note that, for instance, $u_{|M|+3,|M|}$; $u_{|M|+3,|M|+1}$; $u_{|M|+3,|M|+2}$; ... have the same energy $-\frac{\Lambda}{(|M|+3)^2}$, and similarly for other states at the same energy level. It is then evident that given $N \geq |M| + 1$, $u_{N,N-1}$; $u_{N,N-2}$; $u_{N,N-3}$; ...; $u_{N,|M|}$ will all have the same energy $-\frac{\Lambda}{N^2}$. Hence, we can say that

$$E_N = -\frac{\Lambda}{N^2}, \quad \Lambda = \eta^2 \sigma^4 |\epsilon_0|, \quad N = L + 1 + n', \quad n' = 0, 1, 2, \dots \quad (52)$$

which means that the energy is actually labelled by N and not L . Equation 52 agrees with reference [1].

From figure 5, equations 48, 49, 36 and 42, it can be shown that

$$R_{|M|+1,|M|}(r) = \left[\frac{2\gamma}{|M|+1} \right]^{|M|+3/2} \left[\frac{1}{\Gamma(2|M|+3)} \right]^{1/2} r^{|M|} e^{-\gamma r/|M|+1} \quad (53)$$

$$R_{|M|+2,|M|+1}(r) = \left[\frac{2\gamma}{|M|+2} \right]^{|M|+5/2} \left[\frac{1}{\Gamma(2|M|+5)} \right]^{1/2} r^{|M|+1} e^{-\gamma r/|M|+2} \quad (54)$$

$$R_{|M|+2,|M|}(r) = - \left[\frac{2\gamma}{|M|+2} \right]^{|M|+3/2} \left[\frac{1}{2(|M|+2)\Gamma(2|M|+3)} \right]^{1/2} r^{|M|} e^{-\gamma r/|M|+2} \\ \times \left[2|M|+2 - \frac{2\gamma r}{|M|+2} \right] \quad (55)$$

etc. where R_{NL} are normalized by

$$1 = \int_0^\infty |R_{NL}|^2 r^2 dr \quad (56)$$

and the gamma function properties used are [21]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt; \quad \Gamma(z+1) = z\Gamma(z). \quad (57)$$

Equations 53, 54 and 55 agree with the normalized R_{NL} of reference [2]. It is obvious that we can eventually get all the expressions of R_{NL} .

To illustrate the procedure more concretely, let us show how we can get $R_{|M|+2,|M|}(r)$. From figure 5,

$$u_{|M|+2,|M|} \sim A_{|M|}^+ u_{|M|+2,|M|+1}^2 \quad (58)$$

² Note that in equation 2 and in some of the following equations, we use \sim instead of $=$ since the procedure outlined here cannot automatically normalize the eigenfunctions. At the end, we normalize the eigenfunctions using equation 56.

Since $u_{|M|+2,|M|+1}$ is at the lowest rung of $H_{|M|+1}$, we can use equations 48 and 49 which give

$$u_{|M|+2,|M|+1} \sim r^{|M|+2} e^{-\gamma r/[|M|+2]} . \quad (59)$$

From equations 42, 2 and 59, we get

$$u_{|M|+2,|M|} \sim \left(-\frac{d}{dr} - \frac{|M|+1}{r} + \frac{\gamma}{|M|+1} \right) r^{|M|+2} e^{-\gamma r/[|M|+2]} \quad (60)$$

which leads to

$$u_{|M|+2,|M|} \sim r e^{-\gamma r/[|M|+2]} \left(-r^{|M|} + \frac{\gamma r^{|M|+1}}{(|M|+2)(|M|+1)} \right) . \quad (61)$$

Using equations 61 and 36 and rearranging terms, we get

$$R_{|M|+2,|M|} \sim -e^{-\gamma r/[|M|+2]} r^{|M|} \left(2 + 2|M| - \frac{2\gamma r}{|M|+2} \right) \quad (62)$$

or

$$R_{|M|+2,|M|} = -\mathcal{N} e^{-\gamma r/[|M|+2]} r^{|M|} \left(2 + 2|M| - \frac{2\gamma r}{|M|+2} \right) \quad (63)$$

where \mathcal{N} is the normalization constant. Normalizing equation 63 using equations 56 and 57 leads to equation 55.

4 Conclusion

The ring-shaped Hartmann potential was first introduced in 1972 [1]. It has been used to describe ring-shaped molecules like the benzene molecule in theoretical chemistry [2]. In an attempt to introduce the methods of supersymmetric quantum mechanics in quantum chemistry, we obtain the eigenvalues and radial eigenfunctions of the Hartmann potential in spherical coordinates using SUSYQM techniques.

A key result in SUSYQM is the intimate relationship of the eigenvalues and eigenfunctions of the hierarchy of SUSY-partner hamiltonians. This can be very useful in solving the Schrödinger equation of a complicated hamiltonian if its SUSY-partner hamiltonian is easily solvable.

A very useful result in the present discussion is the fact that $A^-\psi^0 = 0$. This enabled us to solve a first order differential equation (as in equation 46) instead of the second order Schrödinger differential equation to give us the eigenfunctions and eigenvalues of the states at the lowest rung of the tower of states of each of the hamiltonians in the hierarchy. The rest of the eigenfunctions and eigenvalues are then easily computed by applying the corresponding A_L^+ operators to these eigenfunctions.

As already indicated in this paper, the separated equations of the Hartmann potential and the hydrogen atom greatly resemble each other. In fact, the technique employed here was first applied to the hydrogen atom [14]. A number of studies of the SUSY features of the Coulomb problem in hydrogenic atoms have been made over the past years [19, 22, 23, 24]. These studies may very well lead to some further insights into the workings of SUSY in the Hartmann potential due to the similarity of its separated equations with that of the hydrogen atom.

The preceding paragraph actually led the author to believe that if the Hartmann problem is treated using the SUSY formulation of Haymaker and Rau [24], the eigenstates of different values of $\eta\sigma^2$ can be related. This will be developed in a future publication.

With the above considerations, the author believes that quantum chemistry is a field where the concepts and techniques of SUSYQM can be put to good use.

Figure Captions

1. The hierarchy of hamiltonians and the action of the operators A_n^\pm on the degenerate eigenstates
2. The energy states of the hierarchy of SUSY-partner hamiltonians from the Hartmann potential
3. The hierarchy of hamiltonians of the Hartmann potential and their ground states. The H_L here are the actual radial hamiltonian for a particular L value.
4. Figure 3 with the energy levels labelled.
5. The energy eigenstates of the Hartmann potential. The action of the A_L^+ operators are explicitly shown to indicate how the other states are obtained from the states at the lowest rung of the hierarchy of hamiltonians.

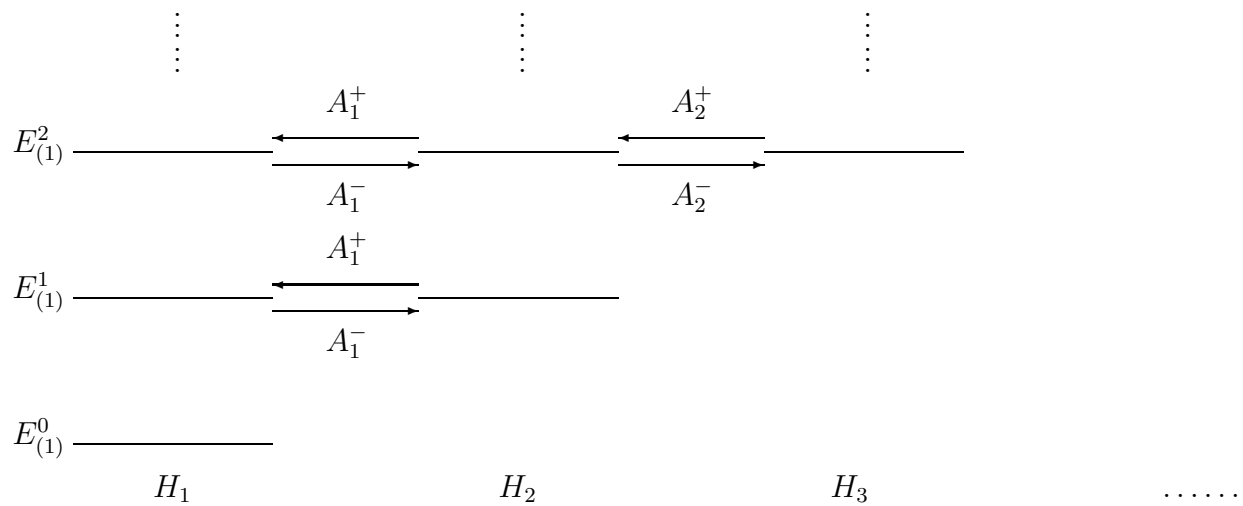


Figure 1: The hierarchy of hamiltonians and the action of the operators A_n^\pm on the degenerate eigenstates

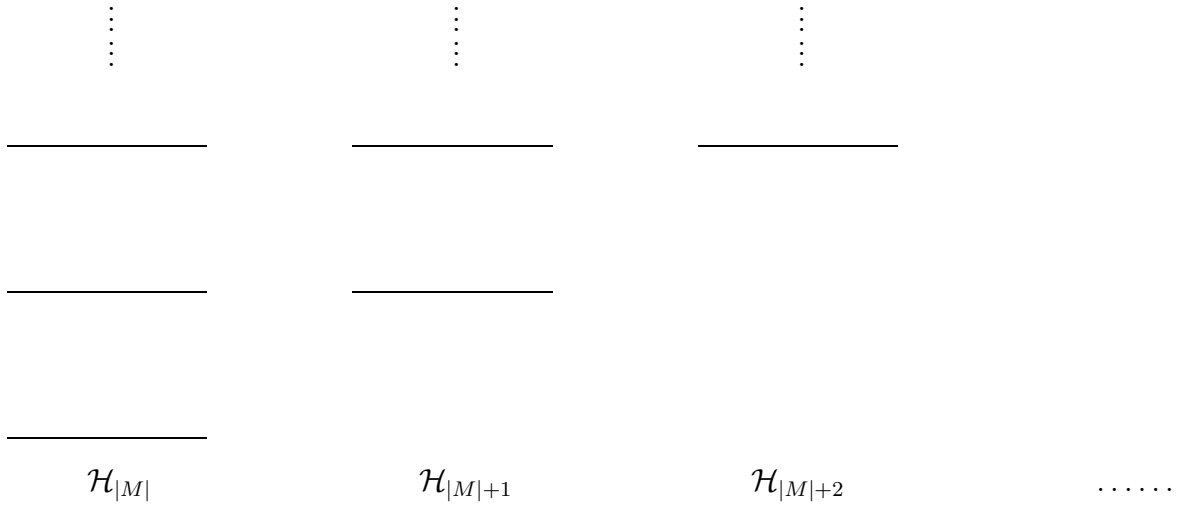


Figure 2: The energy states of the hierarchy of SUSY-partner hamiltonians from the Hartmann potential

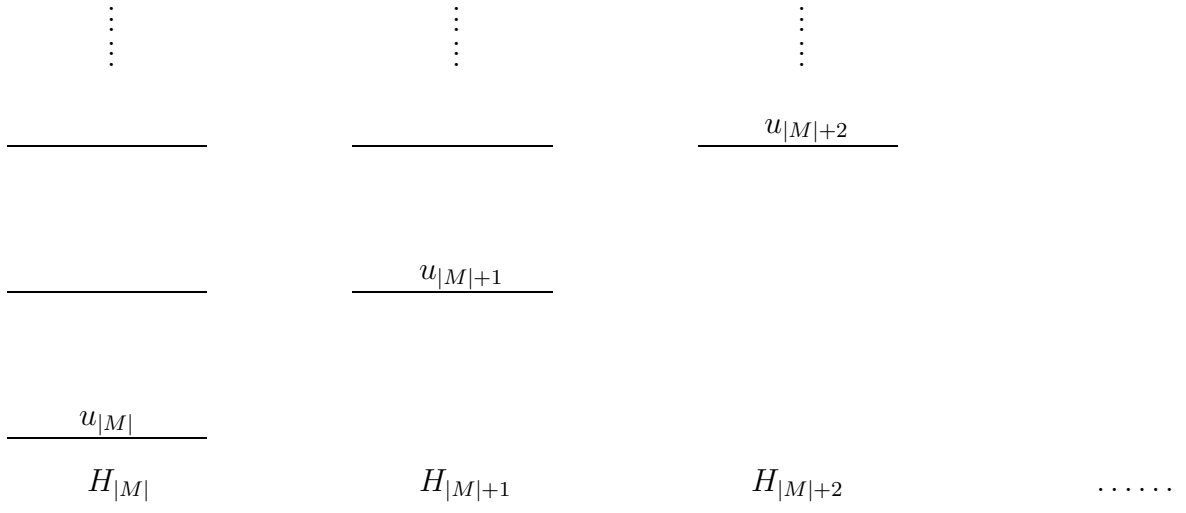


Figure 3: The hierarchy of hamiltonians of the Hartmann potential and their ground states. The H_L here are the actual radial hamiltonian for a particular L value.

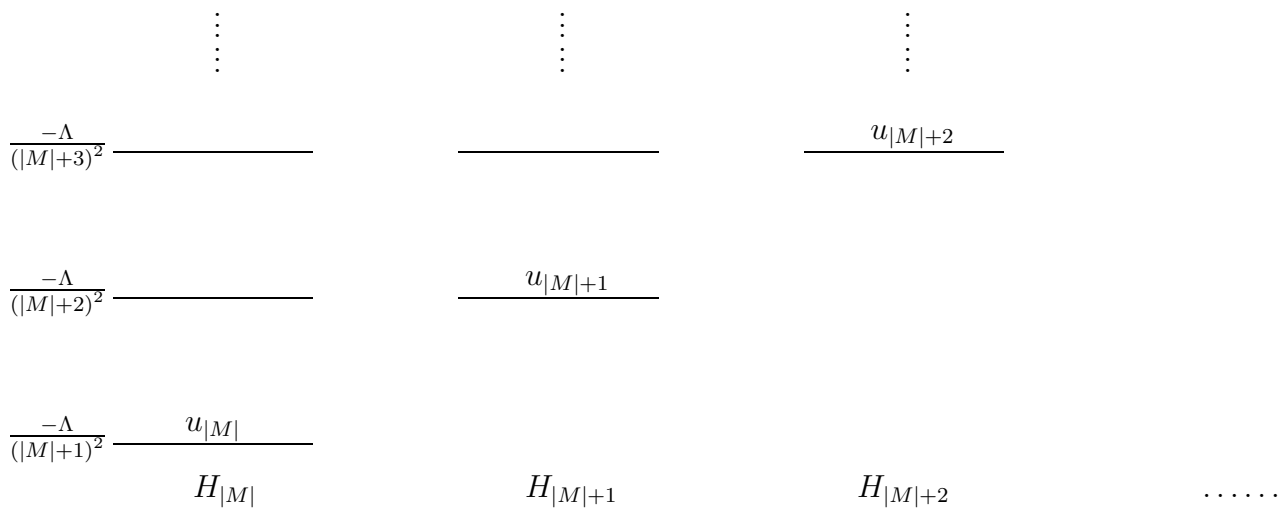


Figure 4: Figure 3 with the energy levels labelled.

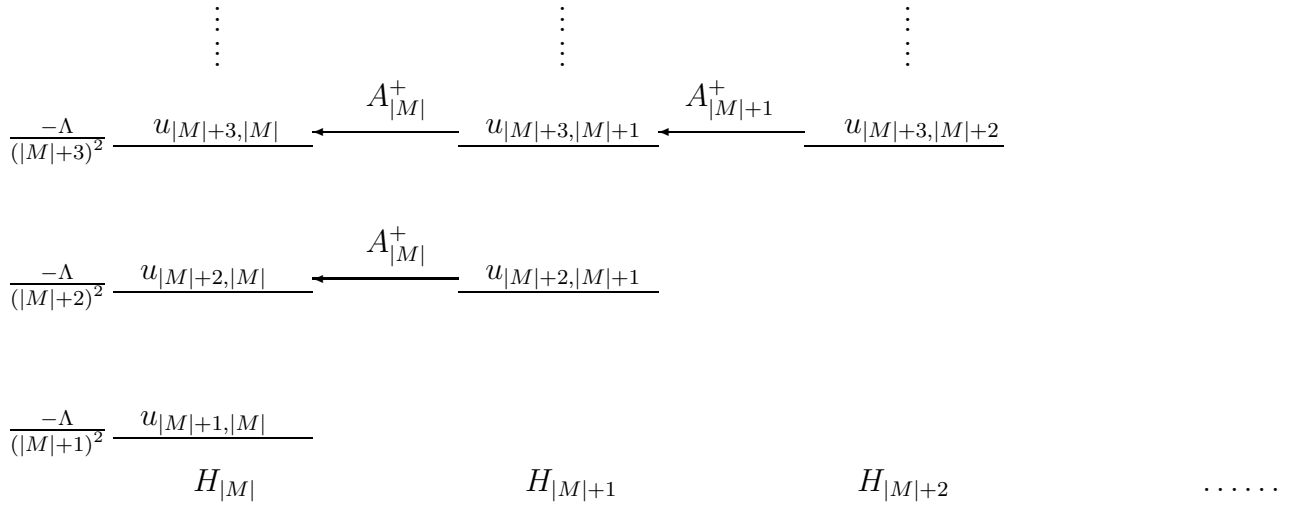


Figure 5: The energy eigenstates of the Hartmann potential. The action of the A_L^+ operators are explicitly shown to indicate how the other states are obtained from the states at the lowest rung of the hierarchy of hamiltonians.

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